Applied Optimization Problems

Problem:- A rectangular garden with 100 ft fencing. Determine the maximum area & the corresponding dimensions.

Solution:- (1) Introduce all variables.

Let x & y be the sides of the rectangular

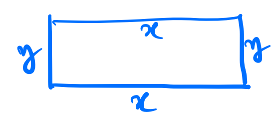
(2) Understand what you are trying to Maximize or Minimize

We want the area, say A, to maximize.

We know formula for area : $A = xy$.

(3) Use constraints to boil down to a function of only one variable.

Here fencing is 100 ft. So, sum of sides is 100ft



So, $x + y + x + y = 100$

$$\Rightarrow 2(x+y) = 100$$

$$\Rightarrow x + y = 50$$

$$\Rightarrow y = 50 - x$$

Then $A = xy = x(50-x)$.

(4) Using Calculus techniques, find max/min of the target quantity.

$$A = x(50-x) = 50x - x^2 \qquad \text{———} \quad (*)$$
$$\overset{\shortparallel}{A(x)}$$

Find critical pts by putting $A'(x) = 0$ or undefined.

Here, $A'(x) = 50 - 2x$, $A''(x) = -2 < 0$

So, $A'(x) = 0 \Rightarrow 50 - 2x = 0 \Rightarrow x = 25$

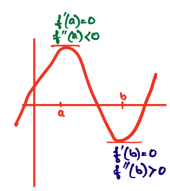
At critical pt, $A(x)$ attains max/min.

So, $A(25) = 50 \times 25 - 25 \times 25$ [by $(*)$].

$$= (50-25) \times 25$$
$$= 25 \times 25$$
$$= 625 \text{ sqft.} \rightsquigarrow \text{Max area.}$$

Note:- At max pt $x=a$, $f'(a) = 0$ & $f''(a) < 0$

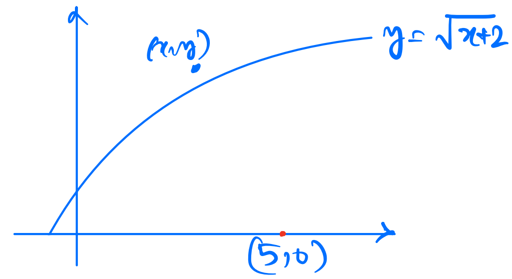
min pt $x=b$, $f'(b) = 0$ & $f''(b) > 0$



Dimensions: $x = 25$ ft, $y = 50 - 25 = 25$ ft.

Problem :- Find the closest point to $(5,0)$ on the curve $y=\sqrt{x+2}$, $x \geq -2$

Sol^n :- Choose a pt on $y=\sqrt{x+2}$. Say (x,y).



Distance between (x,y) & $(5,0)$ is

$$D=\sqrt{(x-5)^2+(y-0)^2}$$
$$=\sqrt{(x-5)^2+y^2}$$

Now, (x,y) is a point on $y=\sqrt{x+2}$, so

$$y=\sqrt{x+2}$$

Then $D=\sqrt{(x-5)^2+y^2}$ can be written in one variable x as:

$$D(x)=\sqrt{(x-5)^2+(\sqrt{x+2})^2}$$
$$=\sqrt{(x-5)^2+(x+2)}$$
$$=\sqrt{x^2-10x+25+x+2}$$
$$=\sqrt{x^2-9x+27}.$$
$$=(x^2-9x+27)^{1/2}$$

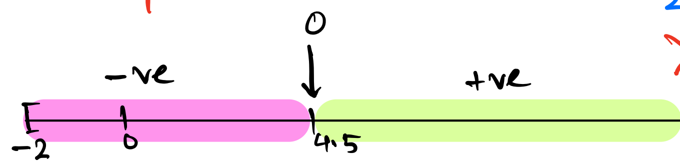
$$D'(x)=\frac{1}{2}(x^2-9x+27)^{-1/2}\cdot(2x-9)$$
$$=\frac{2x-9}{2\sqrt{x^2-9x+27}}$$

$D'(x)=0$ or $D'(x)$ undefined gives

$D'(x) = 0$
$$\Downarrow$$
$2x - 9 = 0$
$\Rightarrow x = 4.5$ critical pt

$D'(x)$ — undefined
$$\Downarrow$$
$x^2 - 9x + 27 = 0$
$\Rightarrow x = \dfrac{9 \pm \sqrt{81 - 4\cdot1\cdot27}}{2\cdot1}$ $\sqrt{-ve}$

Note:- $D'(x)$



with increasing x, $D'(x)$ is increasing
$\Rightarrow D''(x) > 0$
$\Rightarrow D(x)$ has a minima.

So, $x = 4.5 \Rightarrow y = \sqrt{x+2} = \sqrt{6.5}$

Hence, $(4.5, \sqrt{6.5})$ is the closest pt on $y = \sqrt{x+2}$ from $(5, 0)$

Problem:- Find two positive numbers x & y such that $x+2y=50$ & $(x+1)(y+2)$ is maximum.

Sol^n:- Here given $x+2y=50$, then $x=50-2y$

Therefore, maximizing $(x+1)(y+2)$ is same as maximizing $\big((50-2y)+1\big)(y+2)$

$$\text{ie, } (51-2y)(y+2)$$

$$\text{ie } 51y-2y^2+102-4y$$

$$\text{ie } \underbrace{-2y^2+47y+102}_{\text{function of } y}$$

Let $\phi(y)=-2y^2+47y+102$ & we want to find the max value of $\phi(y)$ at some value $y=Y$.

So, we need the critical pts of $\phi(y)$.

Then we take, $\phi'(y)=0$

$$\Rightarrow -4y+47=0$$

$$\Rightarrow y=\frac{47}{4}$$

Also, $\phi''(y)=-4<0 \Rightarrow \phi''\left(\frac{47}{4}\right)<0$

$\Rightarrow \phi$ has a maximum of $y=\frac{47}{4}$

Hence $y = \frac{47}{4}$ & from the condition $x + 2y = 50$

we get $x + 2\left(\frac{47}{4}\right) = 50$

$$\Rightarrow x + \frac{47}{2} = 50$$

$$\Rightarrow x = \frac{53}{2}$$

So the two required pts are $x = \frac{53}{2}$ & $y = \frac{47}{4}$